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Lagrangian and Hamiltonian formalisms for relativistic dynamics of a charged particle with dipole moment

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Abstract. The Lagrangian and Hamiltonian formulations for the relativistic classical dynamics of a charged particle with dipole moment in the presence of an electromagnetic field are given. The differential conservation laws for the energy-momentum and angular momentum tensors of a field and particle are discussed. The Poisson brackets for basic dynamic variables, which form a closed algebra, are found. These Poisson brackets enable us to perform the canonical quantization of the Hamiltonian equations that leads to the Dirac wave equation in the case of spin 1/2. It is also shown that the classical limit of the squared Dirac equation results in equations of motion for a charged particle with dipole moment obtained from the Lagrangian formulation. The inclusion of gravitational field and non-Abelian gauge fields into the proposed formalism is discussed.

1 Introduction

The study of relativistic dynamics of charged particles with internal degrees of freedom (spin) originates from [1, 2]. Later, the equations of motion for such particles in an homogeneous electromagnetic field were derived by generalization of the non-relativistic equations of motion for coordinate and spin of a particle to the relativistic case [3]. Another approach for obtaining the relativistically-invariant equations of motion for charged particles with internal degrees of freedom is discussed by considering complex particles [4]. The definition of the covariant center of energy is the key problem in this approach.

The Lagrangian formulation for the relativistic motion of a charged particle with spin is of special interest. The main advantage of such an approach is that the knowledge of a suitable Lagrangian allows us to obtain results without any additional assumptions. However, the construction of the Lagrangian formalism for charged particles with spin needs the introduction of additional dynamic variables, which are conjugate to the components of relativistic spin described by an antisymmetric tensor [5–9]. Here some authors use singular Lagrangians following the approach suggested by Dirac for constrained Hamiltonian systems [6, 8] (in this case a free Lagrangian is parametrically invariant). However, the alternative formalism based on non-singular Lagrangians is also often used [5, 7, 9] (the free Lagrangian is not parametrically invariant).

The present paper concerns the construction of consistent Lagrangian and Hamiltonian formulations for the relativistic dynamics of charged particles with dipole moment in an electromagnetic field. As a starting point of our consideration we introduce the dipole moment tensor and define the currents associated with a charge of particle and its dipole moment. These definitions are given in accordance with Maxwell's equations. The main idea of the offered Lagrangian approach consists in the introduction of an orthogonal matrix for the rotations in a four-dimensional pseudo-Euclidean space. This matrix determines a fundamental representation of the Lorentz group. It is specified by six independent parameters, which are taken by us to be generalized coordinates. The antisymmetric dipole moment tensor is also characterized by six variables, which play the role of generalized momenta. Using this idea we construct a relativistically-invariant Lagrangian, which leads (in the formalism of proper time) to the relativistically-invariant equations of motion for the basic dynamic variables (four-position and four-momentum of a particle, tensor of dipole moments, and matrix of pseudo-Euclidean rotations).

We also discuss the differential conservation laws for the field and particles and obtain the explicit expressions for the energy-momentum and angular momentum tensors of a charged particle with dipole moments. The analysis of conservation laws enables us to introduce, in the classical case, a tensor of intrinsic angular momentum (spin). The relationship between this tensor and the dipole moment tensor is given by the gyromagnetic ratio expressed

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through the quantities entering the Lagrangian. We show that if the particle has no electric dipole moment, then it possesses a normal magnetic moment that corresponds to the gyromagnetic ratio, equal to e/m.

We would like to note that the introduced matrix of four-rotations has not only a formal but also a simple physical sense: it specifies the evolution of the dipole moment tensor with respect to proper time. This evolution represents a rotation in pseudo-Euclidean space.

We also present the Hamiltonian formulation of the problem. The obtained Hamiltonian of relativistic particles with dipole moments does not contain the above matrix of four-rotations (it represents a cyclic variable). The found Poisson brackets for the basic dynamic variables form a closed algebra. The developed Hamiltonian approach makes it possible to perform the canonical quantization of the obtained Hamiltonian equations. In the case of spin 1/2 this quantization results in the squared Dirac wave equation. We show also that the classical limit of the squared Dirac equation gives the previously obtained equations of motion for a charged particle with dipole moment. In the appendix we generalize the developed formalism to cover motion in gravitational and non-Abelian gauged fields.

2 Relativistically-invariant Lagrangian of point dipole moments

In field theory one can obtain the field equations as well as the equations of motion of particles, by varying an action functional with respect to dynamic variables. In this section we shall construct a relativistically-invariant action functional, which describes the interaction of particles possessing dipole moments with an electromagnetic field.

To begin with, we would like to remind the reader that the four-vector $j_\mu^{(e)}(x)$ of the electric current density created by a particle of charge e is determined by

$$j_\mu^{(e)}(x) = e\int_{-\infty}^{\infty} \mathrm{d}\tau \dot{\xi}_\mu(\tau)\delta(x-\xi(\tau)). \quad (1)$$

Here $\xi_\mu(\tau)$ is the four-trajectory of a particle (τ is a parameter; the dot means derivation with respect to τ; Greek indices take the values $0,1,2,3$). The four-vector (1) of the electric current density satisfies the differential conservation law $\partial^\mu j_\mu^{(e)}(x) = 0$.

If a point particle has dipole moment, then their densities can be specified by means of the antisymmetric tensor $\Sigma^{\mu\nu}(x)$ of rank two

$$\Sigma^{\mu\nu}(x) = \int_{-\infty}^{\infty} \mathrm{d}\tau \sigma^{\mu\nu}(\tau)\delta(x-\xi(\tau)), \quad (2)$$

where $\sigma^{\mu\nu}(\tau)$ is the dipole moment tensor. In addition, the components $\Sigma_{0k}(x) = d_k(x)$ are the densities of the electric dipole moment and $\Sigma_{kl}(x) = \varepsilon_{kls} m_s(x)$ specify the densities of the magnetic dipole moment (see Appendix A). Hence, the four-vector of the current density, which is associated with the dipole moment of the particles is defined by

$$j_\mu^{(d)}(x) = \partial^\nu \Sigma_{\mu\nu}(x) = \partial^\nu \int_{-\infty}^{\infty} \mathrm{d}\tau \sigma_{\mu\nu}(\tau)\delta(x-\xi(\tau)); \quad (3)$$

moreover, $\partial^\mu j_\mu^{(d)}(x) = 0$. The four-vector of the total current density created by a charged particle with dipole moment is given by

$$j_\mu(x) = j_\mu^{(e)}(x) + j_\mu^{(d)}(x), \quad \partial^\mu j_\mu(x) = 0. \quad (4)$$

For particles which have no internal degrees of freedom and interacting with an electromagnetic field the action functional is of the form

$$W = W_\mathrm{f} + W_\mathrm{p} + W_\mathrm{int}, \quad (5)$$

where

$$W_\mathrm{f} = -\frac{1}{16\pi}\int_{t_1}^{t_2} \mathrm{d}^4x F_{\mu\nu}(x)F^{\mu\nu}(x), \quad \mathrm{d}^4x = \mathrm{d}t\mathrm{d}^3x \quad (6)$$

is the part of the action functional that corresponds to the free electromagnetic field $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ ($A_\mu(x)$ is the four-potential),

$$W_\mathrm{int} = -\int_{t_1}^{t_2} \mathrm{d}^4x A_\mu(x) j^\mu(x), \quad (7)$$

describes the interaction of particles with an electromagnetic field, and $W_\mathrm{p} = -m\int_{\tau_1}^{\tau_2}\mathrm{d}\tau\sqrt{\dot{\xi}_\mu\dot{\xi}^\mu}$ is the part of the action functional for free particles (m is the particle mass). Such a form of W_p is parametrically invariant ($\tau \to \tau' = \tau'(\tau)$) and leads to the fact that the generalized momentum $p^\mu = -\partial L_\mathrm{p}/\partial\dot{\xi}_\mu = m\dot{\xi}^\mu/\sqrt{\dot{\xi}_\lambda\dot{\xi}^\lambda}$ (L_p is the Lagrangian corresponding to W_p) lies on the mass shell $p^\mu p_\mu = m^2$. This circumstance does not allow us to consider the p^μ as independent variables when we construct the relativistically-invariant Hamiltonian approach and the corresponding quantum theory (compare to the similar situation in the theory of gauge fields). Therefore, as opposed to the usual theory, we need to break the parametric invariance of W_p. To this end let us choose $W_\mathrm{p} = -\frac{m}{2}\int_{\tau_1}^{\tau_2}\mathrm{d}\tau\dot{\xi}_\mu\dot{\xi}^\mu$. We shall see further that such an action functional results in the correct equations of motion.

However, if a particle has internal degrees of freedom (specified by $\sigma^{\mu\nu}$), then an appropriate part of the action functional should be added to W_p. In order to find it let us draw the analogy to spin variables [10,11]. In the case of spin dynamics three angles $\theta_k(t)$ (the generalized coordinates), which determine the matrix $a_{ik}(\theta)$ of three-dimensional rotations, correspond to three spin variables S_k (the generalized momenta). As a result, the Lagrangian describing the dynamics of free spins is written as $L_0 = -S_i\omega_i$, where $\omega_i = \frac{1}{2}\varepsilon_{ikl}(\tilde{a}(t)\dot{a}(t))_{kl}$ is the so-called Cartan form and $\tilde{a}(t)$ is the transpose matrix $a(t)$.

In the case of the relativistic dynamics of dipole moments, six parameters $\theta_a(t)$ (the generalized coordinates), which determine the orthogonal matrix $a_{\mu\nu}(\theta)$ ($a_{\mu\nu}a^{\mu\lambda} = \delta_\nu^\lambda$) of four-dimensional pseudo-Euclidean rotations, are associated with the six variables $\sigma^{\mu\nu}$ (the generalized momenta). Therefore, using the analogy to spin variables, let us write the Lagrangian corresponding to the dipole moment as $L_0 = -\frac{1}{2\kappa}\sigma^{\mu\nu}a_\mu{}^\lambda\dot{a}_{\nu\lambda}$ (κ is a certain constant, the physical meaning of which we shall clarify below). Thus, we obtain the following action functional for free relativistic particles with dipole moments:

$$W_\mathrm{p} = -\int_{\tau_1}^{\tau_2} \mathrm{d}\tau \left(\frac{m}{2}\dot{\xi}_\mu\dot{\xi}^\mu + \frac{1}{2\kappa}\sigma^{\mu\nu}\omega_{\mu\nu} \right), \tag{8}$$

where $\omega_{\mu\nu} = a_\mu{}^\lambda\dot{a}_{\nu\lambda}$ is the right Cartan form.

Let us transform the expression for W_int. Substitution of (4) into (7) yields

$$W_\mathrm{int} = -\int_{\tau_1}^{\tau_2} \mathrm{d}\tau \left(eA_\mu(\xi)\dot{\xi}^\mu + \frac{1}{2}F_{\mu\nu}(\xi)\sigma^{\mu\nu} \right),$$

where $\xi^0(\tau_1) = t_1$ and $\xi^0(\tau_2) = t_2$. Hence, the total action functional for the particle with dipole moment in an electromagnetic field can be written as

$$\tilde{W} = W_\mathrm{p} + W_\mathrm{int} = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau L(\tau), \tag{9}$$

where the Lagrangian $L(\tau)$ is given by

$$L(\tau) = -\frac{m}{2}\dot{\xi}_\mu\dot{\xi}^\mu - \frac{1}{2\kappa}\sigma^{\mu\nu}a_\mu{}^\lambda\dot{a}_{\nu\lambda} - eA_\mu(\xi)\dot{\xi}^\mu - \frac{1}{2}F_{\mu\nu}(\xi)\sigma^{\mu\nu}. \tag{10}$$

Having this Lagrangian we shall derive the equations of motion for charged particles with dipole moments in an electromagnetic field.

3 Equations of motion for point dipole moments and differential conservation laws

In this section we employ the principle of the stationary action in order to find the equations of motion for the basic dynamic variables. Varying the obtained action functional (9) and (10) with respect to the independent variables ξ^μ, $\sigma^{\mu\nu}$, $a^{\mu\nu}$, and taking the variations of these quantities as zero at the boundaries of the corresponding limits of integration, one finds

$$m\ddot{\xi}^\mu = eF^\mu{}_\nu(\xi)\dot{\xi}^\nu + \frac{1}{2}\sigma^{\nu\lambda}\partial^\mu F_{\nu\lambda}(\xi), \tag{11}$$

$$\dot{\sigma}^{\mu\nu} = (\sigma^{\nu\lambda}a^{\mu\rho} - \sigma^{\mu\lambda}a^{\nu\rho})\dot{a}_{\lambda\rho}, \tag{12}$$

$$\dot{a}^{\mu\nu} = -\kappa F^{\lambda\mu}(\xi)a_\lambda{}^\nu. \tag{13}$$

Let us explain the derivation of (12). The variation of the Lagrangian (10) with respect to $a^{\mu\nu}$ is given by

$$\delta_a L = \frac{1}{2\kappa}\delta a_\mu{}^\lambda(\sigma^{\mu\nu}\dot{a}_{\nu\lambda} - \dot{\sigma}^{\nu\mu}a_{\nu\lambda} - \sigma^{\nu\mu}\dot{a}_{\nu\lambda}).$$

We have omitted the term $(\mathrm{d}/\mathrm{d}\tau)(\sigma^{\mu\nu}\delta a_{\nu\lambda}a_\mu{}^\lambda)$ because it does not contribute to the variation of the action functional. Next using the orthogonality of $a^{\mu\nu}$ ($a_{\rho\lambda}a^{\rho\eta} = \delta_\lambda^\eta$), one gets

$$\delta_a L = \frac{1}{2\kappa}\delta a_\mu{}^\lambda a_{\rho\lambda}(\sigma^{\mu\nu}a^{\rho\eta}\dot{a}_{\nu\eta} - \sigma^{\rho\nu}a^{\mu\eta}\dot{a}_{\nu\eta} - \dot{\sigma}^{\rho\mu}).$$

Since $\delta a_\mu{}^\lambda a_{\rho\lambda}$ is an arbitrary quantity antisymmetric in the indices ρ, μ, the variational principle results in (12).

The obtained relativistically-invariant equations of motion describe the dynamics of a point charged particle with dipole moment in an electromagnetic field. Eliminating $\dot{a}_{\lambda\rho}$ in (12), we get

$$\dot{\sigma}^{\mu\nu} = \kappa\left(\sigma^{\mu\lambda}F^\nu{}_\lambda(\xi) - \sigma^{\nu\lambda}F^\mu{}_\lambda(\xi)\right). \tag{14}$$

Equations (11) and (14) represent the closed system of differential equations for ξ^μ, $\sigma^{\mu\nu}$. Note that (13) can be written as $\omega_{\mu\nu} = -\kappa F_{\mu\nu}$. Thus, the electromagnetic field taken at the point of presence of the particle coincides with the right Cartan form $\omega_{\mu\nu}$ (see its definition above). The dynamic equations (11) and (14) allow us to find the following integrals of motion with respect to the parameter τ:

$$I_1 = m\dot{\xi}^\mu\dot{\xi}_\mu - \sigma^{\mu\nu}F_{\mu\nu}, \quad I_2 = \frac{1}{2}\sigma^{\mu\nu}\sigma_{\mu\nu},$$
$$I_3 = \sigma_{\mu\nu}\sigma_{\lambda\rho}\varepsilon^{\mu\nu\lambda\rho}$$

($\dot{I}_1 = \dot{I}_2 = \dot{I}_3 = 0$). Equations (13) and (14) for $a^{\mu\nu}$ and $\sigma^{\mu\nu}$ give two other integrals of motion,

$$I_4 = \frac{1}{2}a^{\mu\nu}a_{\mu\lambda}, \quad I_5 = a_\mu{}^\lambda a_\nu{}^\rho\sigma^{\mu\nu}, \quad \dot{I}_4 = \dot{I}_5 = 0.$$

Here we have used the antisymmetry of $F^{\mu\nu}$ and $\sigma^{\mu\nu}$.

The integral of motion I_1 determines the proper time parameter τ similar to how $\dot{\xi}_\mu\dot{\xi}^\mu = 1$ determines the proper time in the case of the usual Lorentz equations of motion. The integral of motion I_4 reflects the fact that $a^{\mu\nu}(\tau)$ evolves as an orthogonal matrix. Finally let us give interpretation to I_5. Since the transformation $x_\mu \to x'_\mu = a_{\mu\nu}x^\nu$ preserves the lengths of vectors and angles between them, the relation $\sigma^{\mu\nu}(\tau) = a_\lambda{}^\nu(\tau)a_\rho{}^\mu(\tau)\sigma^{\lambda\rho}(0)$, being a consequence of I_4, shows that in the "spin" space the dipole moment tensor $\sigma^{\mu\nu}(\tau)$ evolves like a "four-solid".

Having the equations of motion we can formulate the differential conservation laws for the energy-momentum and angular momentum tensors. We start from the energy-momentum conservation law. According to the action principle $\delta(W_\mathrm{f} + W_\mathrm{int}) = 0$ (see (6) and (7)), the electromagnetic field $F_{\mu\nu}$ satisfies the Maxwell equations:

$$\partial^\nu F_{\nu\mu} = 4\pi j_\mu, \qquad \partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0, \tag{15}$$

where the total current density $j_\mu(x)$ is given by (1)–(4). These field equations lead to the following formula:

$$\partial_\nu T^{\mu\nu} = -F^{\mu\lambda}j_\lambda, \tag{16}$$

where

$$T^{\mu\nu}(x) = \frac{1}{4\pi}\left(-F^{\mu\rho}F^\nu{}_\rho + \frac{1}{4}g^{\mu\nu}F_{\rho\lambda}F^{\rho\lambda} \right) \tag{17}$$

is the symmetric ($T^{\mu\nu} = T^{\nu\mu}$) energy-momentum tensor of the electromagnetic field and $g^{\mu\nu}$ is the flat space Minkowskian metric. The relation (16) can be transformed into the following differential conservation law:

$$\partial_\nu (T^{\mu\nu} + T'^{\mu\nu}) = 0. \tag{18}$$

To this end we use the simple formula, which follows from (3) and (15),

$$F^{\mu\nu} j_\nu^{(d)} = F^{\mu\nu} \partial^\lambda \Sigma_{\nu\lambda} = \partial^\lambda (F^{\mu\nu} \Sigma_{\nu\lambda}) + \frac{1}{2} \Sigma_{\nu\lambda} \partial^\mu F^{\nu\lambda}.$$

Then bearing in mind also the definition of $\Sigma_{\mu\nu}(x)$ and $j_\nu^{(e)}(x)$ and using (11), one finds

$$F_{\mu\nu} j^\nu = \partial_\lambda \int_{-\infty}^{\infty} \mathrm{d}\tau F_{\mu\nu}(\xi) \sigma^{\nu\lambda}(\tau) \delta(x - \xi(\tau)) + m \int_{-\infty}^{\infty} \mathrm{d}\tau \delta(x - \xi(\tau)) \ddot{\xi}_\mu, \tag{19}$$

whence we come to the differential conservation law (18), where

$$T'^{\mu\nu}(x) = \int_{-\infty}^{\infty} \mathrm{d}\tau t^{\mu\nu}(\tau) \delta(x - \xi(\tau)), \tag{20}$$

$$t^{\mu\nu}(\tau) = F^\mu_{\ \lambda}(\xi) \sigma^{\lambda\nu} + m \dot{\xi}^\mu \dot{\xi}^\nu.$$

The quantity $T'^{\mu\nu}(x)$ should be interpreted as the energy-momentum tensor of a charged particle with dipole moment interacting with an electromagnetic field. Note that $t_\mu^{\ \mu}(\tau) = I_1$. Since the particle proper field diverges on its four-trajectory, the expression for $t^{\mu\nu}$ as well as (11) and (14) require, generally speaking, the renormalization procedure if $F_{\mu\nu}$ is the total electromagnetic field including the particle proper field.

Let us now obtain the differential conservation law for the total relativistic angular momentum tensor. The angular momentum tensor of the electromagnetic field is defined by

$$M^{\mu\nu;\rho}(x) = x^\mu T^{\nu\rho}(x) - x^\nu T^{\mu\rho}(x), \tag{21}$$

whence according to (16), we have

$$\partial_\rho M^{\mu\nu;\rho} = -x^\mu F^{\nu\rho} j_\rho + x^\nu F^{\mu\rho} j_\rho.$$

Similar to the previous calculations this relation can easily be transformed into the differential conservation law for the total angular momentum tensor,

$$\partial_\rho (M^{\mu\nu;\rho} + M'^{\mu\nu;\rho}) = 0. \tag{22}$$

Indeed, noting that

$$\partial_\rho M^{\mu\nu;\rho} = -\partial_\rho \int_{-\infty}^{\infty} \mathrm{d}\tau (\xi^\mu t^{\nu\rho} - \xi^\nu t^{\mu\rho}) \delta(x - \xi(\tau)) + \frac{1}{\kappa} \int_{-\infty}^{\infty} \mathrm{d}\tau \dot{\sigma}^{\nu\mu}(\tau) \delta(x - \xi(\tau))$$

(we have used here (19), (20) and (14)), we come to the differential conservation law (22), where

$$M'^{\mu\nu;\rho}(x) = \int_{-\infty}^{\infty} \mathrm{d}\tau m^{\mu\nu;\rho}(\tau) \delta(x - \xi(\tau)), \tag{23}$$

$$m^{\mu\nu;\rho}(\tau) = \xi^\mu t^{\nu\rho} - \xi^\nu t^{\mu\rho} + I^{\mu\nu} \dot{\xi}^\rho, \quad I^{\mu\nu} = \frac{1}{\kappa} \sigma^{\mu\nu}.$$

The quantity $M'^{\mu\nu;\rho}$ should be interpreted as the angular momentum tensor of a charged particle with dipole moment interacting with an electromagnetic field. Moreover, the first two terms in (23) represent the orbital angular momentum tensor, whereas the third term is the intrinsic (spin) angular momentum tensor. Therefore, the quantity κ should be treated as the gyromagnetic ratio. Note that the quantities $t^{\mu\rho}\dot{\xi}_\rho$, $m^{\mu\nu;\rho}\dot{\xi}_\rho$ specify the four-vector of momentum and four-tensor of the angular momentum of a particle in an electromagnetic field.

In the obtained differential conservation laws (18) and (22), the tensors $T^{\mu\nu}$, $M^{\mu\nu;\rho}$ are defined only by an electromagnetic field. These quantities given by (17) and (21) have an unambiguous interpretation. The tensors $T'^{\mu\nu}$, $M'^{\mu\nu;\rho}$ are associated with particles interacting with an electromagnetic field (see (20) and (23)). The components $T^{\mu 0} + T'^{\mu 0}$ can be treated as the density of energy-momentum ($t^{\mu\nu}$ is not symmetric). Then the first term in $m^{\mu\nu;\rho} = (\xi^\mu t^{\nu\rho} - \xi^\nu t^{\mu\rho}) + I^{\mu\nu}\dot{\xi}^\rho$ determines uniquely the orbital angular momentum, whereas the second one describes the spin angular momentum. In the field formulation, it is always possible to attain the symmetric form of the total tensor $T^{\mu\nu}$. Then $M^{\mu\nu;\rho} = x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}$. However, in this case, it is difficult to extract the orbital angular momentum and spin angular momentum from $M^{\mu\nu;\rho}$. For example, in the monograph [12], only the procedure of the extraction of the spin angular momentum is considered on the basis of the Dirac equation for free particles. Note that the canonical and symmetric tensors of energy-momentum result in the same values of the total energy, momentum, and orbital momentum.

In the usual formulation of general relativity it is necesary to use the symmetric energy-momentum tensor. However, in its modified formulation, which takes into account the effect of torsion, the canonical (not symmetric) energy-momentum tensor should be used [13].

Usually, the elementary particles have only a magnetic dipole moment (the electric dipole moment is absent). This means that the components σ^{0k} are zero in the frame of reference in which a particle is in rest. Since the quantity $\sigma^{\mu\nu}\dot{\xi}_\nu$ represents the four-vector, which is zero in the frame of reference where $\dot{\xi}^\nu = (0, 0, 0, \dot{\xi}^0)$, the relativistically-invariant condition, which reflects the fact of the absence of electric dipole moments, can be written in the form [7,14]

$$\sigma^{\mu\nu} \dot{\xi}_\nu = 0. \tag{24}$$

In order to show the consistency of the constraint (24) with the equations of motion, let us differentiate it with respect to τ and use (11) and (14). As a result, one finds

$$\frac{\mathrm{d}}{\mathrm{d}\tau} (\sigma^{\mu\nu} \dot{\xi}_\nu) = \kappa \sigma^{\lambda\nu} \dot{\xi}_\nu F^\mu_{\ \lambda} + \left(\kappa - \frac{e}{m} \right) \sigma^{\mu\lambda} \dot{\xi}_\nu F^\nu_{\ \lambda}$$

$$+\frac{1}{2m}\sigma^{\mu\nu}\sigma^{\lambda\rho}\partial_\nu F_{\lambda\rho}.$$

Therefore, as it can be easily seen, the constraint (24) is consistent with the dynamic equations (11) and (14) for a homogeneous (slowly varying) field and gyromagnetic ratio $\kappa = e/m$ (the speed of light $c = 1$). The gyromagnetic ratio $\kappa = e/m$ corresponds to the normal magnetic moment. In the general case $\kappa = ge/2mc$ so that the magnetic dipole moment $\mu = \kappa\hbar S$, where g represents the gyromagnetic factor, and S is the spin of a particle. The consistency of (24) with the dynamic equations in the case of a specific inhomogeneous field is discussed in [14].

As we have shown, the obtained set of equations (11) and (14) does not describe the dynamics of particles with an anomalous magnetic moment. This is due to the fact that the requirement $\sigma^{\mu\nu}\dot{\xi}_\nu = 0$ is not consistent with the equations of motion. Therefore, in order to describe the dynamics of particles with an anomalous magnetic moment, we use the method of Lagrangian multipliers. As a result the Lagrangian (10) is of the form

$$L(\tau) = -\frac{m}{2}\dot{\xi}_\mu\dot{\xi}^\mu - \frac{1}{2\kappa}\sigma^{\mu\nu}a_\mu{}^\lambda\dot{a}_{\nu\lambda} - eA_\mu(\xi)\dot{\xi}^\mu$$
$$-\frac{1}{2}F_{\mu\nu}(\xi)\sigma^{\mu\nu} - \varDelta_\mu\sigma^{\mu\nu}\dot{\xi}_\nu, \qquad (25)$$

where the $\varDelta_\mu$ are the Lagrangian multipliers. The variation of the action functional (9) with this Lagrangian over $\varDelta_\mu$, ξ_μ, $\sigma_{\mu\nu}$, $a_{\mu\nu}$ leads to the following set of equations:

$$\sigma^{\mu\nu}\dot{\xi}_\nu = 0, \qquad (26)$$

$$m\ddot{\xi}^\mu + \dot{\eta}^\mu = eF^\mu{}_\nu\dot{\xi}^\nu + \frac{1}{2}\sigma^{\lambda\nu}\partial^\mu F_{\lambda\nu}, \qquad (27)$$

$$\dot{\sigma}^{\mu\nu} = (\sigma^{\nu\lambda}a^{\mu\rho} - \sigma^{\mu\lambda}a^{\nu\rho})\dot{a}_{\lambda\rho}, \qquad (28)$$

$$\dot{a}_{\mu\nu} = -\kappa a^\lambda{}_\nu(F_{\lambda\mu} - \dot{\xi}_\lambda\varDelta_\mu + \dot{\xi}_\mu\varDelta_\lambda), \qquad (29)$$

where $\eta^\mu = -\sigma^{\mu\nu}\varDelta_\nu$; moreover, $\eta^\mu\dot{\xi}_\mu = 0$. The elimination of $\dot{a}_{\lambda\rho}$ in (28) results in the following equation for $\sigma^{\mu\nu}$:

$$\dot{\sigma}^{\mu\nu} = \kappa(\sigma^{\mu\lambda}F^\nu{}_\lambda - \sigma^{\nu\lambda}F^\mu{}_\lambda + \dot{\xi}^\nu\eta^\mu - \dot{\xi}^\mu\eta^\nu). \qquad (30)$$

The quantity $\eta^\mu = -\sigma^{\mu\nu}\varDelta_\nu$ entering the closed system of dynamic equations (27) and (30) can be found as a result of multiplication of (30) by $\dot{\xi}_\nu$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\sigma^{\mu\nu}\dot{\xi}_\nu) = \sigma^{\mu\nu}\ddot{\xi}_\nu$$
$$+\kappa(\sigma^{\mu\lambda}F^\nu{}_\lambda\dot{\xi}_\nu - \sigma^{\nu\lambda}F^\mu{}_\lambda\dot{\xi}_\nu + \dot{\xi}^\nu\dot{\xi}_\nu\eta^\mu - \dot{\xi}^\mu\dot{\xi}_\nu\eta^\nu)$$

whence taking into account the requirement (26) and the definition of η^μ, one finds

$$\eta^\mu = \frac{(\ddot{\xi}_\nu - \kappa\dot{\xi}_\lambda F_\nu{}^\lambda)\sigma^{\nu\mu}}{\kappa\dot{\xi}^\nu\dot{\xi}_\nu}, \quad \kappa = \frac{ge}{2m}. \qquad (31)$$

Note that the dynamic equations (27) and (30) agree with those derived in [5, 7]. However, the authors of [5] used the requirement $\sigma^{\mu\nu}(p_\mu - eA_\mu) = 0$ (p_μ is the canonical momentum) instead of $\sigma^{\mu\nu}\dot{\xi}_\nu = 0$. As we shall show the latter requirement follows from the classical limit of the squared Dirac equation. The dynamic equations (27) and (30) along with (31) lead to the following relation, which determines the parameter τ:

$$\dot{\xi}_\mu\dot{\xi}^\mu = 1 + \frac{1}{m}\sigma^{\nu\lambda}F_{\nu\lambda}.$$

Let us show that (27) and (30) are in correspondence with Bargmann–Michel–Telegdi (BMT) equations in the case of weak and homogeneous electromagnetic fields. In doing so, we introduce the Pauli–Lyubanskii spin vector,

$$w^\mu = \frac{1}{2\kappa}\epsilon^{\mu\nu\lambda\rho}\sigma_{\nu\lambda}\dot{\xi}_\rho, \qquad (32)$$

whence

$$\sigma_{\mu\nu} = \kappa\epsilon_{\mu\nu\lambda\rho}\dot{\xi}^\lambda w^\rho. \qquad (33)$$

Using (30), (32) and (33), one finds

$$\dot{w}^\mu = \kappa(\dot{\xi}^\mu\dot{\xi}_\nu w^\lambda F_\lambda{}^\nu - \dot{\xi}^\nu\dot{\xi}_\nu w^\lambda F_\lambda{}^\mu) + (\dot{\xi}^\nu w^\mu - \dot{\xi}^\mu w^\nu)\ddot{\xi}_\nu.$$

Here we have employed the fact that $\dot{\xi}_\mu w^\mu = 0$. In the linear approximation in the field $F_{\mu\nu}$ and zeroth approximation in the gradients of the electromagnetic field, the latter equation along with (27) take the form

$$m\ddot{\xi}^\mu = eF^\mu{}_\nu\dot{\xi}^\nu, \qquad (34)$$
$$\dot{w}^\mu = \left(\kappa - \frac{e}{m}\right)\dot{\xi}^\mu\dot{\xi}_\nu w^\lambda F_\lambda{}^\nu - \kappa w^\nu F_\nu{}^\mu,$$
$$\kappa = \frac{ge}{2m}. \qquad (35)$$

We have neglected $\dot{\eta}^\mu$ because this quantity contains the gradients of an electromagnetic field and the terms nonlinear in $F_{\mu\nu}$. The obtained equations (34) and (35) represent the BMT equations [3].

In conclusion of this section we would like to note that in the weak-field limit and with small inhomogeneities the quantity η^μ, according to (27) and (31) is defined as

$$\eta^\mu = \frac{1}{\kappa}\left(\frac{e}{m} - \kappa\right)F_\nu{}^\lambda\dot{\xi}_\lambda\sigma^{\nu\mu}.$$

In the case of a normal magnetic moment ($\kappa = e/m$), η^μ becomes zero and (27) and (20) coincide with the equations of motion (11) and (14).

4 The Hamiltonian approach

We are now in a position to construct the Hamiltonian approach and to obtain the Poisson brackets for basic dynamic variables. Following the conventional rules let us introduce the canonical four-momentum p^μ, which can be written according to (10) as

$$p^\mu = -\frac{\partial L}{\partial\dot{\xi}_\mu} = m\dot{\xi}^\mu + eA^\mu(\xi). \qquad (36)$$

Then the Lagrangian (10) becomes $L = L_{\mathrm{k}} - H$, where L_{k} is its kinematic part linear with respect to derivatives of the dynamic variables ξ^μ, $a_{\mu\nu}$ and H is the Hamiltonian,

$$L_{\mathrm{k}} = -p^\mu \dot{\xi}_\mu - \frac{1}{2\kappa}\sigma^{\mu\nu} a_\mu{}^\lambda \dot{a}_{\nu\lambda}, \tag{37}$$

$$H = -\frac{1}{2m}(p^\mu - eA^\mu(\xi))(p_\mu - eA_\mu(\xi)) + \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}(\xi). \tag{38}$$

Here ξ^μ, p^μ, $a^{\mu\nu}$, $\sigma^{\mu\nu}$ are the dynamic variables; moreover, $a_{\mu\nu}$ is a cyclic variable (it does not enter the Hamiltonian (38)).

The Poisson brackets are determined by the kinematic part structure of the Lagrangian. The matrix $a_\mu{}^\lambda(\theta)$ depends on six parameters (four-rotations), which are conjugate to the six variables $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$. Let us denote them through θ_a $(a = 1, \dots 6)$ assuming $\theta_a = \theta_{\mu\nu}$ at $\mu < \nu$ and $\theta_{\mu\nu} = -\theta_{\nu\mu}$ at $\mu > \nu$. As a result the kinematic part of the Lagrangian is of the form

$$L_{\mathrm{k}} = -p^\mu \dot{\xi}_\mu - \frac{1}{2}\chi^{\mu\nu}\dot{\theta}_{\mu\nu}, \tag{39}$$

where

$$\chi^{\mu\nu} = \frac{1}{2\kappa}\sigma^{\lambda\rho}R^{\mu\nu}_{\lambda\rho}, \quad R^{\mu\nu}_{\lambda\rho} = a_\lambda{}^\sigma \frac{\partial a_{\rho\sigma}}{\partial\theta_{\mu\nu}}. \tag{40}$$

As it can be easily seen from the structure of (39), the non-trivial Poisson brackets have the form

$$\{p^\mu, \xi_\nu\} = \delta^\mu_\nu, \quad \{\chi^{\mu\nu}, \theta_{\lambda\rho}\} = \delta^{(\mu\nu)}_{(\lambda\rho)} \equiv \delta^\mu_\lambda\delta^\nu_\rho - \delta^\nu_\lambda\delta^\mu_\rho. \tag{41}$$

Having obtained the second expression from (41) we are able to find the Poisson brackets for $\sigma^{\mu\nu}$, $a^{\lambda\rho}$. First of all, since $\{\theta_{\mu\nu}, \theta_{\lambda\rho}\} = 0$,

$$\{a_{\mu\nu}, a_{\lambda\rho}\} = 0. \tag{42}$$

Next, noting that (see (40) and (41))

$$\delta^{(\mu\nu)}_{(\lambda\rho)} = \frac{1}{2\kappa}R^{\mu\nu}_{\eta\sigma}\{\sigma^{\eta\sigma}, \theta_{\lambda\rho}\} = \frac{1}{2\kappa}\{\sigma^{\mu\nu}, \theta_{\eta\sigma}\}R^{\eta\sigma}_{\lambda\rho}, \tag{43}$$

or

$$\frac{1}{2\kappa}\{\sigma^{\mu\nu}, \theta_{\eta\sigma}\}a_\lambda{}^\kappa \frac{\partial a_{\rho\kappa}}{\partial\theta_{\eta\sigma}} = \delta^\mu_\lambda\delta^\nu_\rho - \delta^\nu_\lambda\delta^\mu_\rho,$$

one obtains

$$\frac{1}{\kappa}\{\sigma^{\mu\nu}, a_{\rho\sigma}\} = a^\mu{}_\sigma\delta^\nu_\rho - a^\nu{}_\sigma\delta^\mu_\rho. \tag{44}$$

In order to find $\{\sigma^{\mu\nu}, \sigma^{\lambda\rho}\}$ we employ the fact that $\{\chi^a, \chi^b\} = 0$ with $\chi^a = (1/\kappa)R^a_b\sigma^b$ $(b = \mu\nu$ at $\mu < \nu)$. Then taking into account (42) we have

$$0 = \frac{1}{\kappa^2}\{R^a_c\sigma^c, R^b_d\sigma^d\} = \frac{1}{\kappa^2}\left(R^a_cR^b_d\{\sigma^c, \sigma^d\} + R^b_d\sigma^c\{R^a_c, \sigma^d\} + R^a_c\sigma^d\{\sigma^c, R^b_d\}\right).$$

The use of (43) results in

$$R^a_c\{\sigma^c, R^b_d\}\sigma^d = R^a_c\{\sigma^c, \theta_l\}\frac{\partial R^b_d}{\partial\theta_l}\sigma^d = \kappa\frac{\partial R^b_d}{\partial\theta_a}\sigma_d.$$

Therefore,

$$\frac{1}{\kappa}R^a_cR^b_d\{\sigma^c, \sigma^d\} + \left(\frac{\partial R^b_d}{\partial\theta_a} - \frac{\partial R^a_d}{\partial\theta_b}\right)\sigma^d = 0.$$

Noting that

$$\frac{\partial R^b_d}{\partial\theta_a} - \frac{\partial R^a_d}{\partial\theta_b} = \frac{\partial a_\mu{}^\lambda}{\partial\theta_a}\frac{\partial a_{\nu\lambda}}{\partial\theta_b} - \frac{\partial a_\mu{}^\lambda}{\partial\theta_b}\frac{\partial a_{\nu\lambda}}{\partial\theta_a} = g^{\rho\sigma}R^a_{\rho\mu}R^b_{\sigma\nu} - g^{\rho\sigma}R^a_{\rho\nu}R^b_{\sigma\mu}, \quad d \equiv \mu\nu,$$

we obtain

$$\frac{1}{\kappa}R^a_cR^b_d\{\sigma^c, \sigma^d\} = -g^{\rho\sigma}R^a_{\rho\mu}R^b_{\sigma\nu}\sigma^{\mu\nu}.$$

The right-hand side of this expression should be reduced to an antisymmetric form with respect to the indices ρ, μ and σ, ν (in doing so we need to bear in mind that $R^a_{\rho\mu} = -R^a_{\mu\rho}$). The result is

$$\frac{1}{\kappa}\{\sigma^{\sigma\mu}, \sigma^{\kappa\nu}\} = g^{\mu\kappa}\sigma^{\sigma\nu} - g^{\sigma\kappa}\sigma^{\mu\nu} - g^{\mu\nu}\sigma^{\sigma\kappa} + g^{\sigma\nu}\sigma^{\mu\kappa}. \tag{45}$$

The Poisson brackets (44) and (45) can be written in terms of $I^{\mu\nu} = (1/\kappa)\sigma^{\mu\nu}$:

$$\{I^{\mu\nu}, a_{\rho\kappa}\} = a^\mu{}_\kappa\delta^\nu_\rho - a^\nu{}_\kappa\delta^\mu_\rho, \tag{46}$$
$$\{I^{\sigma\mu}, I^{\kappa\nu}\} = g^{\mu\kappa}I^{\sigma\nu} - g^{\sigma\kappa}I^{\mu\nu} - g^{\mu\nu}I^{\sigma\kappa} + g^{\sigma\nu}I^{\mu\kappa}.$$

Note that the second Poisson brackets looks like the commutation relation for the infinitesimal operators of the Lorentz group (see the next section).

The Hamiltonian equations of motion for the basic dynamic variables are of the form

$$\dot{\xi}^\mu = \{\xi^\mu, H\}, \quad \dot{p}^\mu = \{p^\mu, H\},$$
$$\dot{\sigma}^{\mu\nu} = \{\sigma^{\mu\nu}, H\}, \quad \dot{a}^{\mu\nu} = \{a^{\mu\nu}, H\}. \tag{47}$$

It is easy to prove that these equations coincide with (11)–(13).

5 Quantization of the equations of motion for point dipole moments

In this section applying the Dirac procedure, we consider the quantization of (11)–(13) written in the Hamiltonian form (see (47)). The quantization for the spin variables on the basis of the Schwinger dynamic principle [15] is discussed in [16].

When constructing the quantum theory, the dynamic variables ξ^μ, p^μ, $I^{\mu\nu}$ should be replaced by the corresponding operators $\hat{\xi}^\mu$, $\hat{p}^\mu$, $\hat{I}^{\mu\nu}$ (these variables form a closed algebra, whereas $a^{\mu\nu}$ is a cyclic variable). Next, in order to obtain the equations of motion for these operators, we need (according to Dirac) to replace in (47) the Poisson brackets by commutators $\{\ldots, H\} \to -\mathrm{i}[\ldots, \hat{H}]$ (the Hamiltonian is given by (38) and $I^{\mu\nu} = \sigma^{\mu\nu}/\kappa$). As a result the equations of motion in Heisenberg's representation with respect to the parameter τ have the form

$$\dot{\hat{p}}^\mu = -\mathrm{i}[\hat{p}^\mu, \hat{H}], \quad \dot{\hat{\xi}}^\mu = -\mathrm{i}[\hat{\xi}^\mu, \hat{H}], \quad \dot{\hat{I}}^{\mu\nu} = -\mathrm{i}[\hat{I}^{\mu\nu}, \hat{H}]; \tag{48}$$

moreover, the dynamic variables meet the following nontrivial equal-time commutation relations (see (41) and (46)):

$$-\mathrm{i}[\hat{p}^\mu, \hat{\xi}_\nu] = \delta^\mu_\nu, \tag{49}$$

$$-\mathrm{i}[\hat{I}^{\lambda\mu}, \hat{I}^{\kappa\nu}] = g^{\mu\kappa}\hat{I}^{\lambda\nu} - g^{\lambda\kappa}\hat{I}^{\mu\nu} + g^{\lambda\nu}\hat{I}^{\mu\kappa} - g^{\mu\nu}\hat{I}^{\lambda\kappa}.$$

In addition, the first one coincides with the permutation relation for the generators of the Lorentz group. In this representation, the state vector ψ does not depend on τ, $i\frac{\partial\psi}{\partial\tau} = 0$.

In the Schroedinger representation with respect to the parameter τ, the above operators do not depend on τ, whereas the state vector $\psi(\tau)$ satisfies the Schroedinger equation $i\frac{\partial\psi(\tau)}{\partial\tau} = \hat{H}\psi(\tau)$. Let us choose its stationary solution: $\psi(\tau) = \mathrm{e}^{-\mathrm{i}\epsilon\tau}\psi$,

$$\hat{H}\psi = \epsilon\psi. \tag{50}$$

This solution corresponds to the particle "energy" ϵ when the Hamiltonian H does not depend on τ, $\epsilon = H|_{\tau\to-\infty} = -m/2$ (we consider that a particle is out of the field action at $\tau \to -\infty$; moreover, $\dot{\xi}^\mu\dot{\xi}_\mu = 1$). Thus, taking into account (38), (50) takes the form

$$\Big\{(\hat{p}^\mu - eA^\mu(\hat{\xi}))(\hat{p}_\mu - eA_\mu(\hat{\xi})) - e\hat{I}^{\mu\nu}F_{\mu\nu}(\hat{\xi}) - m^2\Big\}\psi(\hat{\xi}) = 0. \tag{51}$$

Let us take the realization of $\hat{\xi}^\mu$, $\hat{p}^\mu$ by means of the multiplication operators x^μ and the differential operators $\mathrm{i}\partial^\mu$,

$$\hat{\xi}^\mu \to x^\mu, \qquad \hat{p}^\mu \to \mathrm{i}\partial^\mu.$$

If we now take the realization of $\hat{I}^{\mu\nu}$ as $\hat{I}^{\mu\nu} = 0$ in (51), then we come to the Klein–Gordon equation in the presence of an electromagnetic field, which corresponds to scalar particles,

$$\left\{(\partial^\mu + \mathrm{i}eA^\mu(x))(\partial_\mu - \mathrm{i}eA_\mu(x)) + m^2\right\}\psi(x) = 0.$$

In order to obtain the realization of the operators $\hat{I}^{\mu\nu}$ for spin equal 1/2 we should seek the solution for $\hat{I}^{\mu\nu}$ in the class of double-row matrices. This solution is of the form

$$\hat{I}^{\mu\nu} = \frac{\mathrm{i}}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu), \tag{52}$$

where $\sigma^\mu \equiv (\sigma^0 = 1, \sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices and $\bar{\sigma}^\mu = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3)$. Therefore, in the considered representation of the dynamic variables, the wave function ψ is a two-component function of x, $\psi \to \psi_\alpha(x)$. It satisfies the squared Dirac equation [17]:

$$\Big\{(\partial^\mu + \mathrm{i}eA^\mu)(\partial_\mu + \mathrm{i}eA_\mu) + e\hat{I}^{\mu\nu}F_{\mu\nu} + m^2\Big\}\psi(x) = 0. \tag{53}$$

Noting that

$$\frac{1}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu) = \bar{\sigma}^\mu\sigma^\nu - g^{\mu\nu},$$

this equation can be written, according to (52), in the form

$$\left\{m^2 + \bar{\sigma}^\mu\sigma^\nu(\partial_\mu + \mathrm{i}eA_\mu)(\partial_\nu + \mathrm{i}eA_\nu)\right\}\psi(x) = 0.$$

Introducing next a new field $\chi(x)$,

$$\mathrm{i}\sigma^\nu(\partial_\nu + \mathrm{i}eA_\nu)\psi(x) = m\chi(x), \tag{54}$$

we can represent the latter equation as

$$\mathrm{i}\bar{\sigma}^\mu(\partial_\mu + \mathrm{i}eA_\mu)\chi(x) = m\psi(x). \tag{55}$$

If now we introduce

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \qquad \phi = \begin{pmatrix} \psi \\ \chi \end{pmatrix},$$

then (54) and (55) take the form of the Dirac equation in the Weyl representation,

$$(\mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}eA_\mu(x)) - m)\phi(x) = 0, \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

Thus, one can think that the realization of $\hat{I}^{\mu\nu}$ by the matrices of higher rank leads to the relativistically-invariant equations for higher spins because of

$$\hat{I}^{\mu\nu} \to \hat{I}'^{\mu\nu} = a^\mu{}_\lambda a^\nu{}_\rho \hat{I}^{\lambda\rho}.$$

However, we shall not consider here this possibility. Note only that the minimal coupling with the electromagnetic field is replaced by the following requirement: the wave function ψ should satisfy the gauge-invariant equation (51) in the presence of an electromagnetic field, where $\hat{I}^{\mu\nu}$ is realized by the matrices of higher rank.

6 The classical limit of the squared Dirac equation

In this section we shall show how (11) and (14) can be obtained from the Dirac wave equation as a result of the transition to the classical limit. In particular, we shall look for the approximate solution of (53), which becomes zero out of a classical trajectory $\xi_\mu(\tau)$ for $\hbar \to 0$. For these solutions, the bilinear combinations $\bar{\psi}_\alpha(x)\psi_\beta(x)$ should have the δ-function behavior over $\boldsymbol{x} - \boldsymbol{\xi}(t)$. Such a behavior for the solutions of wave equations (wave packets) is valid at times $0 < t < t_0$, where t_0 is a duration of spreading for a wave packet ($t_0 \to \infty$ at $\hbar \to 0$).

Let us demonstrate the above mentioned by considering a simple example of non-relativistic Schroedinger equation for a free particle. If the wave function at the moment of time $t = 0$ represents the following wave packet of width σ:

$$\psi(x, 0) = (\pi\sigma)^{-1/4} \exp\left\{\frac{\mathrm{i}p_0 x}{\hbar} - \frac{x - x_0}{2\sigma}\right\},$$

then the solution of the Schroedinger equation in the coordinate representation has the form

$$\begin{aligned}\psi(x,t) &\equiv \psi_\xi(x) \\ &= \frac{(\pi\sigma)^{-1/4}}{\left(1+\frac{\mathrm{i}\hbar t}{m\sigma}\right)^{1/2}} \\ &\quad\times \exp\left\{\frac{\mathrm{i}p_0 x}{\hbar} - \frac{\mathrm{i}p_0^2}{2m\hbar}t - \frac{(x-\xi(t))^2}{2\sigma\left(1+\frac{\mathrm{i}\hbar t}{m\sigma}\right)}\right\},\end{aligned}$$

where p_0 and x_0 are the particle momentum and space coordinate at $t = 0$, and $\xi(t) = x_0 + v_0 t$, $p_0 = mv_0$. The wave function in the momentum representation is given by

$$\begin{aligned}&\psi(p,t) \\ &= \left(\frac{\sigma}{\pi\hbar^2}\right)^{1/4} \\ &\quad\times \exp\left\{-\frac{\mathrm{i}p^2}{2m\hbar}t - \frac{\mathrm{i}(p-p_0)x_0}{\hbar} - \frac{(p-p_0)^2}{2\hbar}\sigma\right\}.\end{aligned}$$

For $t \ll \frac{m\sigma}{\hbar} = t_0$ we have

$$|\psi(x,t)|^2 = (\pi\sigma)^{-1/2}\exp\left\{-\frac{(x-\xi(t))^2}{\sigma}\right\},$$

$$|\psi(p,t)|^2 = \left(\frac{\sigma}{\pi\hbar^2}\right)^{1/2}\exp\left\{-\frac{(p-p_0)^2}{\hbar^2}\sigma\right\};$$

moreover, $|\psi(x,t)|^2 \xrightarrow[\sigma\to 0]{} \delta(x-\xi(t))$ and $|\psi(p,t)|^2 \xrightarrow[\hbar\to 0]{} \delta(p-p_0)$. Note that in the classical limit, the wave function $\psi_\xi(x)$ is an eigenfunction of the non-commuting operators for space coordinate and momentum,

$$\begin{aligned}\hat{x}\psi_\xi(x) &= x\psi_\xi(x) = \xi\psi_\xi(x), \\ \hat{p}\psi_\xi(x) &= -\mathrm{i}\frac{\partial}{\partial x}\psi_\xi(x) = p_0\psi_\xi(x).\end{aligned} \tag{56}$$

This fact does not contradict the commutation relation $[x, p] = \mathrm{i}\hbar$ (in the limit of $\hbar \to 0$ the operators $\hat{x}$ and $\hat{p}$ do commute).

We assume that in the case of relativistic wave equations the described above situation is also true. The action functional that results in the squared Dirac equation (53) has the form

$$W = \int_{t_1}^{t_2} \mathrm{d}^4x\, L(x), \tag{57}$$

$$L(x) = \frac{1}{2m}\bar{\psi}\overleftarrow{D}_\mu\overrightarrow{D}^\mu\psi - \frac{e}{2m}\bar{\psi}\hat{I}^{\mu\nu}F_{\mu\nu}\psi - \frac{m}{2}\bar{\psi}\psi,$$

where $\overleftarrow{D}_\mu = \overleftarrow{\partial}_\mu - \mathrm{i}eA_\mu$, $\overrightarrow{D}_\mu = \overrightarrow{\partial}_\mu + \mathrm{i}eA_\mu$ are the covariant derivatives and $\bar{\psi}(x) = \psi^*(x)\gamma^0$.

In order to obtain the dynamic equations (11) and (14) as the classical limit of the squared Dirac equation, we consider the following functional R_a:

$$R_a = \int \mathrm{d}^3x \left(\frac{\partial L}{\partial\psi} - \partial_\mu\frac{\partial L}{\partial\partial_\mu\psi}\right)B_a\psi. \tag{58}$$

Here $B_a = \{1,\ \hat{I}^{\mu\nu},\ x^l,\ \mathrm{i}D^\mu\}$ are the operators acting on the field index and space coordinate. Let us now introduce the indefinite scalar product

$$(\psi_1, \psi_2) = \int \mathrm{d}^3x\bar{\psi}_1(x)\psi_2(x).$$

In this scalar product, the Hermitian conjugate operation is defined in the standard way:

$$(\psi_1, A\psi_2) = (A^\dagger\psi_1, \psi_2),$$

where $A^\dagger = \gamma^0 A^+\gamma^0$ (A^+ is the Hermitian conjugate operation in the usual scalar product $\int \mathrm{d}^3x\psi_1^*(x)\psi_2(x)$). Note that all the introduced operators B_a are Hermitian, $B_a = B_a^\dagger$. It can easily be shown that for two arbitrary operators A and C, the following formula is valid:

$$(\psi_1, AC\psi_2)^* = (\psi_2, C^\dagger A^\dagger\psi_1). \tag{59}$$

Therefore, the functional R_a^* is of the form

$$R_a^* = \int \mathrm{d}^3x\bar{\psi}B_a\left(\frac{\partial L}{\partial\bar{\psi}} - \partial_\mu\frac{\partial L}{\partial\partial_\mu\bar{\psi}}\right). \tag{60}$$

The Lagrangian L is real and bilinear over $\psi(x)$ and $\bar{\psi}(x)$; see for instance (57). The functionals (58) and (60) go to zero if $\psi(x)$ and $\bar{\psi}(x)$ satisfy the Euler–Lagrange equation, which results in the squared Dirac equation. However, in our next calculations we shall assume that these functions do not satisfy the Euler–Lagrange equation.

It is evident that (58) can be written in the form

$$\begin{aligned}R_a = &-\partial_0\int \mathrm{d}^3x\frac{\partial L}{\partial\partial_\mu\psi}B_a\psi \\ &+\int \mathrm{d}^3x\left(\frac{\partial L}{\partial\psi} + \frac{\partial L}{\partial\partial_\mu\psi}\partial_\mu\right)B_a\psi\ .\end{aligned} \tag{61}$$

Next, noting that the Lagrangian (57) gives

$$\begin{aligned}\frac{\partial L}{\partial\partial_\mu\psi} &= \frac{1}{2m}\bar{\psi}\overleftarrow{D}^\mu, \\ \frac{\partial L}{\partial\psi} &= -\frac{m}{2}\bar{\psi} - \frac{e}{2m}\bar{\psi}\hat{I}^{\mu\nu}F_{\mu\nu} + \frac{1}{2m}\mathrm{i}eA^\mu\bar{\psi}\overleftarrow{D}_\mu,\end{aligned}$$

one finds

$$\begin{aligned}R_a = &-\frac{1}{2m}\partial_0\int \mathrm{d}^3x\bar{\psi}\overleftarrow{D}^0 B_a\psi \\ &+\frac{1}{2m}\int \mathrm{d}^3x\bar{\psi}\left(\overleftarrow{D}_\mu\overrightarrow{D}^\mu - e\hat{I}^{\mu\nu}F_{\mu\nu} - m^2\right)B_a\psi.\end{aligned}$$

Then, according to (60), the functional R_a^* is of the form

$$R_a^* = -\frac{1}{2m}\partial_0 \int \mathrm{d}^3x\bar{\psi}B_a\overrightarrow{D}^0\psi + \frac{1}{2m}\int \mathrm{d}^3x\bar{\psi}B_a\left(\overleftarrow{D}_\mu\overrightarrow{D}^\mu - e\hat{I}^{\mu\nu}F_{\mu\nu} - m^2\right)\psi.$$

These two formulae result in

$$\begin{aligned} R_a - R_a^* &\equiv b_a \\ &= -\frac{1}{2m}\partial_0\int \mathrm{d}^3x\bar{\psi}\left(\overleftarrow{D}^0B_a - B_a\overrightarrow{D}^0\right)\psi \\ &+\frac{1}{2m}\int \mathrm{d}^3x\bar{\psi}\left(e[B_a,\hat{I}^{\mu\nu}F_{\mu\nu}]\right)\psi \\ &+\frac{1}{2m}\int \mathrm{d}^3x\bar{\psi}\left(\overleftarrow{D}^\mu\left[\overrightarrow{D}_\mu, B_a\right] - \left[B_a,\overleftarrow{D}_\mu\right]\overrightarrow{D}^\mu\right)\psi. \end{aligned}$$

In a similar manner, one obtains

$$R_a + R_a^* = b_a - c_a,$$

where

$$c_a = \frac{1}{m}\int \mathrm{d}^3x\bar{\psi}B_a\left(\overleftarrow{D}_\mu\overrightarrow{D}^\mu + e\hat{I}^{\mu\nu}F_{\mu\nu} + m^2\right)\psi.$$

Here we have employed the fact that $\overleftarrow{D}_\mu = -\overrightarrow{D}_\mu + \overleftarrow{\partial}_\mu + \overrightarrow{\partial}_\mu$. If $\psi(x)$ satisfies the Euler–Lagrange equation with the Lagrangian (57) resulting in the squared Dirac equation (53), then $R_a = R_a^* = 0$, $c_a = 0$, and consequently $b_a(t) = 0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x\bar{\psi}\left(\overleftarrow{D}^0B_a - B_a\overrightarrow{D}^0\right)\psi \quad (62) \\ = \int \mathrm{d}^3x\bar{\psi}\left(e[B_a,\hat{I}^{\mu\nu}F_{\mu\nu}] + \overleftarrow{D}^\mu\left[\overrightarrow{D}_\mu, B_a\right]\right. \\ \left. - \left[B_a,\overleftarrow{D}_\mu\right]\overrightarrow{D}^\mu\right)\psi. \end{aligned}$$

In the case of $B_a = \mathrm{i}\overrightarrow{D}_k$, (62) should be derived separately. The result is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x\bar{\psi}\left(\overleftarrow{D}^0\overrightarrow{D}_k + \overleftarrow{D}_k\overrightarrow{D}^0\right)\psi \quad (63) \\ = \int \mathrm{d}^3x\bar{\psi}\left(e\left[\overrightarrow{D}_k,\hat{I}^{\mu\nu}F_{\mu\nu}\right]\right)\psi \\ + \int \mathrm{d}^3x\bar{\psi}\left(\overleftarrow{D}^\mu\left[\overrightarrow{D}_\mu,\overrightarrow{D}_k\right] + \left[\overleftarrow{D}_k,\overleftarrow{D}_\mu\right]\overrightarrow{D}^\mu\right)\psi \end{aligned}$$

(here we have taken into account that $\overrightarrow{D}_\mu + \overleftarrow{D}_\mu = \overrightarrow{\partial}_\mu + \overleftarrow{\partial}_\mu$).

As well as in the non-relativistic case, let us assume that the functions $\psi_\alpha(x)$, $\bar{\psi}_\beta(x)$ represent the wave packets different from zero only for $|\boldsymbol{x}-\boldsymbol{\xi}(t)| \ll \sigma^{1/2}$ and $0 < t < t_0$ (t_0 is a spreading time of the wave packet, which tends to infinity at $\hbar \to 0$; σ is the width of the packet). Therefore, approximately (the smaller $\hbar$ and σ, the more exactly), the functions $\psi_\alpha(x)$, $\bar{\psi}_\beta(x)$ are the eigenfunctions of the non-commuting operators $\mathrm{i}D^\mu$ and x^l,

$$\begin{aligned} \mathrm{i}\overrightarrow{D}^\mu\psi(x) &\approx \pi^\mu(t)\psi(x), \\ \mathrm{i}\bar{\psi}(x)\overleftarrow{D}^\mu &\approx -\pi^\mu(t)\bar{\psi}(x), \\ x^l\psi(x) &\approx \xi^l(t)\psi(x), \quad \bar{\psi}(x)x^l \approx \xi^l(t)\bar{\psi}(x) \end{aligned} \quad (64)$$

(compare to (56)).

If we calculate the commutators in the right-hand side of (62), then it can be easily seen that the terms linear in $\hat{I}^{\mu\nu}$ have the following structure:

$$Q^{\mu\nu}(t) = \int \mathrm{d}^3x\bar{\psi}_\alpha(x)A(x)\hat{I}^{\mu\nu}_{\alpha\beta}C(x)\psi_\beta(x),$$

where $A(x)$, $C(x)$ are the operators constructed of x^l, $\mathrm{i}D^\mu$. Let us consider now the calculation of such terms. If we were in the classical limit, $\psi_\beta(x)$ were the eigenfunctions of the matrices $\hat{I}^{\mu\nu}_{\alpha\beta}$ (as the operators $\mathrm{i}\overrightarrow{D}^\mu$, x^l),

$$\hat{I}^{\mu\nu}_{\alpha\beta}\psi_\beta(x) = I^{\mu\nu}(t)\psi_\alpha(x), \quad (65)$$

then the calculation of $Q^{\mu\nu}(t)$ should be trivial. However, $\hat{I}^{\mu\nu}_{\alpha\beta}$ are finite-row matrices (with respect to the indices α and β), which do not commute and, consequently, (65) is not valid. Therefore, we should act in another way. Namely, the quantity $Q^{\mu\nu}(t)$ can be written in the form

$$Q^{\mu\nu}(t) = A(\xi^l(t))C(\xi^l(t))\int \mathrm{d}^3x\bar{\psi}_\alpha(x)\hat{I}^{\mu\nu}_{\alpha\beta}\psi_\beta(x). \quad (66)$$

We have employed here the fact that $\bar{\psi}_\alpha(x)$, $\psi_\beta(x)$ differs from zero only for $x^l \approx \xi^l$. Let us define $I^{\mu\nu}(t)$ by the formula

$$I^{\mu\nu}(t)\int \mathrm{d}^3x\bar{\psi}_\alpha(x)\psi_\alpha(x) = \int \mathrm{d}^3x\bar{\psi}_\alpha(x)\hat{I}^{\mu\nu}_{\alpha\beta}\psi_\beta(x), \quad (67)$$

which reflects the fact that $I^{\mu\nu}(t)$ is the "average" value of $\hat{I}^{\mu\nu}_{\alpha\beta}$ in the state $\psi(x)$. Then, (66) is of the form

$$Q^{\mu\nu}(t) = A(\xi^l(t))C(\xi^l(t))I^{\mu\nu}(t)\int \mathrm{d}^3x\bar{\psi}_\alpha(x)\psi_\alpha(x). \quad (68)$$

This relationship will be used in calculation of the right-hand side of (62).

We are now in a position to obtain the equations of motion for point relativistic particles with dipole moments in an electromagnetic field. Setting $B_a = 1$ in (62) and using (64), one finds

$$\pi^0(t)\chi(t) = \mathrm{const}, \quad \chi(t) = \int \mathrm{d}^3x\bar{\psi}_\alpha(x)\psi_\alpha(x). \quad (69)$$

Let now $B_a = x^l$. Then, noting that

$$\left[\overrightarrow{D}_\mu, x^l\right] = \left[x^l, \overleftarrow{D}_\mu\right] = \delta^l_\mu,$$

one obtains from (62)

$$\frac{\mathrm{d}\xi^l}{\mathrm{d}t} = \frac{\pi^l}{\pi^0}$$

(we have taken into account (64) and (69)). Next, let us define the proper time of a particle by the formula

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\pi^0}{m}. \tag{70}$$

Then

$$m\dot{\xi}^\mu = \pi^\mu. \tag{71}$$

In order to find the dynamic equation for $I^{\mu\nu}$ ($B_a = \hat{I}^{\mu\nu}$ in (62)) we note that

$$[\hat{I}^{\lambda\rho}, \hat{I}^{\mu\nu}F_{\mu\nu}] = 2\mathrm{i}F_{\sigma\kappa}(g^{\rho\sigma}\hat{I}^{\lambda\kappa} + g^{\lambda\kappa}\hat{I}^{\rho\sigma}),$$
$$\left[\overrightarrow{D}_\mu, \hat{I}^{\lambda\rho}\right] = \left[\hat{I}^{\lambda\rho}, \overleftarrow{D}_\mu\right] = 0.$$

Therefore, according to (64), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi^0(t)\int \mathrm{d}^3x\bar{\psi}(x)\hat{I}^{\lambda\rho}\psi(x)$$
$$= e\int \mathrm{d}^3x\bar{\psi}(x)(g^{\rho\sigma}\hat{I}^{\lambda\kappa} + g^{\lambda\kappa}\hat{I}^{\rho\sigma})F_{\sigma\kappa}\psi(x),$$

whence bearing in mind (67)–(69), one gets

$$\frac{\mathrm{d}I^{\mu\nu}}{\mathrm{d}t} = \frac{e}{\pi^0(t)}F_{\lambda\rho}(\xi)(g^{\nu\lambda}I^{\mu\rho} + g^{\mu\rho}I^{\nu\lambda}).$$

Performing here the transition to the differentiation over τ in accordance with (70), we find finally

$$\dot{I}^{\mu\nu} = \frac{e}{m}F_{\lambda\rho}(\xi)(g^{\nu\lambda}I^{\mu\rho} + g^{\mu\rho}I^{\nu\lambda}). \tag{72}$$

Let us now obtain the equation of motion for π^μ. To this end we address (63), which is an analogue of (62) for $B_a = \mathrm{i}\overrightarrow{D}_k$. It is easy to find the commutators entering the right-hand side of this equation,

$$\left[\overrightarrow{D}_k, \hat{I}^{\mu\nu}F_{\mu\nu}\right] = \hat{I}^{\mu\nu}\partial_k F_{\mu\nu},$$
$$\left[\overrightarrow{D}_\mu, \overrightarrow{D}_\nu\right] = \left[\overleftarrow{D}_\mu, \overleftarrow{D}_\nu\right] = \mathrm{i}eF_{\mu\nu}.$$

Performing the calculations similar to those that were done in deriving (72), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_k(t) = \frac{e}{2\pi^0(t)}I^{\mu\nu}\partial_k F_{\mu\nu}(\xi) - \frac{e}{\pi^0(t)}\pi^\mu F_{\mu k}(\xi),$$

or according to (70),

$$\dot{\pi}_k = \frac{e}{2m}I^{\mu\nu}\partial_k F_{\mu\nu}(\xi) - e\dot{\xi}^\mu F_{\mu k}(\xi). \tag{73}$$

Here we have used (71). To reduce this equation to the relativistically-invariant form, let us return again to the squared Dirac equation (53). Upon multiplying it by $\bar{\psi}$ and integrating over d^3x, one finds

$$\pi_\mu\pi^\mu = eF_{\mu\nu}I^{\mu\nu} + m^2.$$

The differentiation of this relation with respect to τ leads to

$$2\pi_0\dot{\pi}^0 + 2\pi_k\dot{\pi}^k = eI^{\mu\nu}\partial_\lambda F_{\mu\nu}\dot{\xi}^\lambda.$$

(in virtue of (14) $F_{\mu\nu}\dot{I}^{\mu\nu} = 0$). On the other hand, according to (73) we have

$$\pi^k\dot{\pi}_k = \frac{e}{2m}I^{\mu\nu}\pi^k\partial_k F_{\mu\nu} - e\pi^k\dot{\xi}^0 F_{0k}.$$

Therefore,

$$2\pi_0\dot{\pi}^0 = eI^{\mu\nu}(\partial_0 F_{\mu\nu})\dot{\xi}^0 + 2e\pi^k\dot{\xi}^0 F_{0k},$$

whence taking into account that $\pi^0 = m\dot{\xi}^0$, one obtains

$$\dot{\pi}_0 = \frac{e}{2m}I^{\mu\nu}\partial_0 F_{\mu\nu} + e\dot{\xi}^\nu F_{0\nu}.$$

Hence, this equation and (73) combine into the following relativistically-invariant equation:

$$m\ddot{\xi}_\rho = \dot{\pi}_\rho = eF_{\rho\nu}\dot{\xi}^\nu + \frac{e}{2m}I^{\mu\nu}\partial_\rho F_{\mu\nu}. \tag{74}$$

The derived equation along with (72) describes the dynamics of a charged particle with dipole moment in an electromagnetic field. The obtained equations (72) and (74) completely coincide with (11) and (14), which were derived on the basis of the variational principle for the corresponding Lagrangian. Within other approaches, the classical limit of the Dirac equation was considered in [14,18].

In conclusion of this section we would like to emphasize that the dynamic equations (72) and (74), which have been derived from the squared Dirac equation as a result of the passage to the classical limit, are consistent with the requirement $\sigma^{\mu\nu}\dot{\xi}_\nu = 0$ (the absence of electric dipole moment).

7 Conclusion

In this paper we have presented Lagrangian and Hamiltonian formalisms for the relativistic dynamics of charged particles with dipole moments (electric and magnetic) in the presence of an electromagnetic field. In order to describe these internal degrees of freedom, we have introduced the dipole moment tensor $\sigma^{\mu\nu}$ and the orthogonal matrix $a_{\mu\nu}$ of four-rotations in the pseudo-Euclidean space, which is conjugate to the dipole moment tensor. The obtained equations of motion agree with the results of other authors. We have also defined a dipole current through the dipole moment tensor. This current has allowed us to formulate the differential conservation laws and to find the explicit expressions for the energy-momentum and angular momentum tensors of the considered particles. The analysis of the differential conservation laws has resulted (in a natural way) in definitions of a relativistic spin as the intrinsic mechanical angular momentum and gyromagnetic ratio. The latter is expressed in terms of the constants entering the Lagrangian. It has been shown that if a particle has no electric dipole moment (in this case $\sigma^{\mu\nu}\dot{\xi}_\nu = 0$), then the gyromagnetic ratio κ is equal to e/m, which corresponds to the normal magnetic moment. The introduced matrix $a_{\mu\nu}$ has not only a formal but also a simple physical sense: it completely specifies the evolution of the dipole moments:

$$\sigma^{\mu\nu}(\tau) = a^{\mu}{}_{\lambda}(\tau)a^{\nu}{}_{\rho}(\tau)\sigma^{\lambda\rho}(0).$$

We have found the Poisson brackets for the basic dynamic variables. These Poisson brackets have been essentially used in the quantization of the obtained equations of motion. The canonical quantization procedure has led to the squared Dirac equation. In the last section we have formulated a method for obtaining the classical equations of motion from the wave equations. It is based on the consideration of the localized wave packets, which determine a particle trajectory. Applying this method to the squared Dirac equation, we have come to those dynamic equations discussed within the Lagrangian approach. The inclusion of the gravitational field and non-Abelian gauge fields associated with the $SU(n)$ group into the proposed Lagrangian formulation has been considered.

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Appendix A: The dipole moment tensor

Let us show that $\Sigma^{\mu\nu}(x)$ specifies the densities of the electric and magnetic dipole moments. To this end we address the Maxwell equations,

$$\partial^\nu F_{\mu\nu} = -4\pi j_\mu, \quad \partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0\,, \quad \text{(A.1)}$$

where $j_\mu(x)$ is defined by (1)–(4). The first equation from (A.1) can be written in the form

$$\partial^\nu(F_{\mu\nu} + 4\pi\Sigma_{\mu\nu}) = -4\pi j_\mu^{(e)}. \quad \text{(A.2)}$$

Setting here $\mu = 0$ and $\nu = k$ one obtains

$$\frac{\partial}{\partial x^k}(E_k + 4\pi\Sigma_{0k}) = 4\pi j_0^{(e)}, \quad E_k = F_{0k},$$

whence the quantity $E_k(x) + 4\pi\Sigma_{0k}(x)$ should be interpreted as the electric displacement vector $D_k(x)$ and $\Sigma_{0k}(x)$ as the density of the electric dipole moment $P_k(x) \equiv d_k(x)$,

$$d_k(x) \equiv \Sigma_{0k}(x) = \int_{-\infty}^{\infty} \mathrm{d}\tau\sigma^{0k}(\tau)\delta(x - \xi(\tau)).$$

Performing here the integration over τ we have

$$d_k(\boldsymbol{x}) = \frac{\sigma_{0k}(\tau)}{\dot{\xi}_0(\tau)}\delta(\boldsymbol{x} - \boldsymbol{x}(t)), \quad t = \xi_0(\tau).$$

Thus, the quantity $\sigma_{0k}(\tau)/\dot{\xi}_0(\tau)$ represents the electric dipole moment of a moving particle.

Let us now demonstrate that $\Sigma_{kl}(x)$ specifies the density of the magnetic dipole moment. For this purpose we set $\mu = l$ in (A.2),

$$\partial^0(F_{l0} + 4\pi\Sigma_{l0}) + \partial^k(F_{lk} + 4\pi\Sigma_{lk}) = -4\pi j_l^{(e)}.$$

Since $F_{l0} + 4\pi\Sigma_{l0} = -E_k - 4\pi P_k$ and $F_{lk} = -\varepsilon_{lks}B_s$ (B_s is real magnetic field),

$$-\frac{\partial}{\partial x^k}(-\varepsilon_{lks}B_s + 4\pi\Sigma_{lk}) = \frac{\partial D_l}{\partial t} + 4\pi j_l^{(e)}.$$

The comparison of this equation to $\operatorname{rot}\boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + 4\pi\boldsymbol{j}^{(e)}$ results in

$$-\varepsilon_{lks}B_s + 4\pi\Sigma_{lk} = -\varepsilon_{lks}H_s,$$

whence

$$H_s = B_s - 4\pi\frac{1}{2}\varepsilon_{slk}\Sigma_{lk}.$$

Therefore, the quantity $\frac{1}{2}\varepsilon_{slk}\Sigma_{lk}$ should be identified with the magnetic moment density $M_s \equiv m_s(x)$,

$$m_s(x) = \frac{1}{2}\varepsilon_{slk}\Sigma_{lk}(x) = \frac{1}{2}\varepsilon_{slk}\int_{-\infty}^{\infty}\mathrm{d}\tau\sigma_{lk}(\tau)\delta(x - \xi(\tau)),$$

or

$$m_s(x) = \frac{1}{2}\varepsilon_{slk}\frac{\sigma_{lk}(\tau)}{\dot{\xi}_0(\tau)}\delta(\boldsymbol{x} - \boldsymbol{x}(t)), \quad t = \xi_0(\tau).$$

Hence, $\sigma_{23}(\tau)/\dot{\xi}_0(\tau)$, $\sigma_{31}(\tau)/\dot{\xi}_0(\tau)$, and $\sigma_{12}(\tau)/\dot{\xi}_0(\tau)$ are the x, y, z components of the magnetic dipole moment of a moving particle.

Appendix B: Equations of motion in gravitational and non-Abelian gauge fields

Here we are concerned briefly with the generalization of the developed approach to the case of a charged point particle with dipole moment interacting not only with an electromagnetic field but also with a gravitational and non-Abelian gauge fields. The possibility of including non-Abelian gauge fields was discussed in [19]. The starting point of our approach is the following Lagrangian:

$$L = L_\mathrm{k} - H, \quad \text{(B.1)}$$

with its kinematic part L_k (this part determines the Poisson brackets for the basic dynamic variables) and Hamiltonian H (with respect to the proper time τ),

$$L_\mathrm{k} = -p_\mu\dot{\xi}^\mu - \frac{1}{2}I^{ik}a_i{}^s\dot{a}_{ks} + 2\mathrm{i}q^a\mathrm{Sp}T_a\dot{u}u^+, \quad \text{(B.2)}$$

$$H = -\frac{1}{2m}\pi^\mu\pi_\mu + \frac{1}{2m}I^{\mu\nu}\mathcal{F}_{\mu\nu}. \quad \text{(B.3)}$$

Here

$$\pi_\mu = p_\mu - eA_\mu - gq_a A^a_\mu - \frac{1}{2} I_{ik} A^{ik}_\mu, \tag{B.4}$$

$$\mathcal{F}_{\mu\nu} = eF_{\mu\nu} + gq_a F^a_{\mu\nu} + \frac{1}{2} I^{ik} R_{ik\mu\nu},$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{bc}{}^a A^b_\mu A^c_\nu \tag{B.5}$$

($R_{ik\mu\nu}$ is the curvature tensor). The non-Abelian charge q_a ($a = 1, \ldots n^2-1$) specifies the interaction of "colored quarks" and defines the currents $J^\mu_a(x)$, which enter the Yang–Mills field equations,

$$\mathcal{D}_\mu F^{\nu\mu}_a = gJ^\nu_a, \qquad J^\mu_a(x) = \int_{-\infty}^{\infty} \mathrm{d}\tau q_a(\tau)\dot{\xi}^\mu \delta(x - \xi(\tau)).$$

The quantities A_μ, A^a_μ, and A^{ik}_μ are respectively the electromagnetic, $SU(n)$, and Lorentz connections. The unitary matrix u with $\det u = 1$ (see (B.2)) depends on n^2-1 "generalized coordinates", which are conjugate to the n^2-1 "generalized momenta" q_a. T_a are the generators of the fundamental representation of the $SU(n)$ group.

In order to explain the structure of the third term in (B.2) we consider some properties of the quantity $\omega(U) = \omega^+(U)$ ($U \in SU(n)$),

$$\omega(U) = \mathrm{i}U\dot{U}^+, \quad \dot{U} = \frac{\mathrm{d}U}{\mathrm{d}t}.$$

Let us introduce the notation

$$\mathcal{R}_a(U) = \mathrm{Sp} T_a \omega(U).$$

Let $v \in SU(n)$. Then, if v does not depend on t, then

$$\mathcal{R}_a(vU) = \mathrm{Sp} T_a v \omega(U) v^+.$$

Noting that the Hermitian matrix $v^+ T_a v$ with zero trace can be expanded in the generators T_b of the fundamental representation of the $SU(n)$ group,

$$v^+ T_a v = u_a{}^b(v) T_b,$$

we have

$$\mathcal{R}_a(vU) = u_a{}^b(v)\mathcal{R}_b(U),$$

where $u_a{}^b(v)$ is the orthogonal matrix. Thus, under the transformations $U \to vU$, the quantity $\mathcal{R}_a(U)$ is transformed as a $SU(n)$ vector in the adjoined representation. If q_a is also transformed as a vector, then $q^a\mathcal{R}_a$ is a scalar. Therefore, the quantity $\mathrm{i}q^a \mathrm{Sp} T_a U\dot{U}^+$ is invariant with respect to the $SU(n)$ transformations and can be used in the construction of the Lagrangian for free particles. Similarly, one can prove that the second term in the Lagrangian (B.2) is invariant with respect to four-rotations.

As dynamic variables in the Lagrangian L we choose the following pairs: (p_μ, ξ^μ); (a_{ks}, I^{ik}); (q_a, u). It is clear that the Poisson brackets among the dynamic variables belonging to different pairs are equal to zero. The nontrivial Poisson brackets inside of the pairs are determined, according to (B.2), by

$$\{p_\mu, \xi^\nu\} = \delta^\nu_\mu, \quad \{I^{ik}, a_{ls}\} = a^i{}_s \delta^k_l - a^k{}_s \delta^i_l,$$
$$\{I^{ik}, I^{ls}\} = \eta^{kl} I^{is} + \eta^{is} I^{kl} - \eta^{il} I^{ks} - \eta^{ks} I^{il}, \tag{B.6}$$
$$\{q_a, q_b\} = f_{ab}{}^c q_c, \quad \{u, q_a\} = -\mathrm{i}T_a u,$$

where η^{ik} is the flat space Minkowskian metric.

In the tetrahedral formalism, the Lorentz tensors $A^{l\cdots}$ can be related to the world tensors $A^{\mu\cdots}$ by $A^{l\cdots} = b^l{}_\mu A^{\mu\cdots}$, where $b^l{}_\mu(x)$ is a tetrahedral field. Now the metric tensor is defined by $g_{\mu\nu}(x) = b^l{}_\mu(x) b_{l\nu}(x)$. The Lorentz connection A^μ_{ik} is related to the Christoffel symbols and field $b^l{}_\mu(x)$ by the formula

$$\Gamma^\mu_{\nu\lambda} = b_l{}^\mu (\partial_\lambda b^l{}_\nu + A_{k\lambda}^l b^k{}_\nu).$$

As a result the following Poisson brackets are obtained from (B.4)–(B.6):

$$\{I^{\mu\nu}, I^{\lambda\rho}\} = g^{\nu\lambda} I^{\mu\rho} + g^{\mu\rho} I^{\nu\lambda} - g^{\mu\lambda} I^{\nu\rho} - g^{\nu\rho} I^{\mu\lambda},$$
$$\{\pi_\mu, \pi_\nu\} = -eF_{\mu\nu} - gq_a F^a_{\mu\nu} - \frac{1}{2} I^{\lambda\rho} R_{\lambda\rho\mu\nu},$$
$$\{\pi_\mu, \xi^\nu\} = \delta^\nu_\mu, \qquad \{\pi_\mu, I^{\nu\lambda}\} = I^{\lambda\rho}\Gamma^\nu_{\rho\mu} - I^{\nu\rho}\Gamma^\lambda_{\rho\mu},$$
$$\{\pi_\mu, q_a\} = g f_{ab}{}^c A^b_\mu q_c, \qquad \{q_a, q_b\} = f_{ab}{}^c q_c.$$

We do not write the Poisson brackets containing the variables a_{ik} and u because these variables, being cyclic ones, do not enter the Hamiltonian.

The equations of motion for the dynamic variables are obtained from the Hamiltonian equations $\dot{r} = \{r, H\}$ (r is a dynamic variable) and have the form

$$\begin{aligned}
&\ddot{\xi}^\rho + \Gamma^\rho_{\lambda\sigma}\dot{\xi}^\lambda\dot{\xi}^\sigma \\
&= \frac{1}{m} g^{\rho\alpha}\dot{\xi}^\mu \left(eF_{\alpha\mu} + gq_a F^a_{\alpha\mu} + \frac{1}{2} I^{\gamma\kappa} R_{\gamma\kappa\alpha\mu}\right) \\
&\quad + \frac{1}{2m^2} g^{\rho\alpha} I^{\mu\nu} \\
&\quad \times \left(e\mathcal{D}_\alpha F_{\mu\nu} + gq_a \mathcal{D}_\alpha F^a_{\mu\nu} + \frac{1}{2} I^{\gamma\kappa}\mathcal{D}_\alpha R_{\gamma\kappa\mu\nu}\right),
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
&\dot{I}^{\mu\nu} + \dot{\xi}^\rho I^{\mu\lambda}\Gamma^\nu_{\lambda\rho} + \dot{\xi}^\rho I^{\lambda\nu}\Gamma^\mu_{\lambda\rho} \\
&= \frac{1}{m}(g^{\nu\lambda} I^{\mu\rho} - g^{\mu\lambda} I^{\nu\rho})(eF_{\lambda\rho} + gq_a F^a_{\lambda\rho} + I^{\sigma\eta} R_{\sigma\eta\lambda\rho}),
\end{aligned} \tag{B.8}$$

$$\dot{q}_a - g f_{ab}{}^c A^b_\rho q_c \dot{\xi}^\rho = \frac{g}{2m} f_{ab}{}^c q_c I^{\mu\nu} F^b_{\mu\nu}. \tag{B.9}$$

The covariant derivative $\mathcal{D}_\alpha$ is defined by

$$\mathcal{D}_\alpha G^a_{\mu\nu} = \partial_\alpha G^a_{\mu\nu} - G^a_{\mu\lambda}\Gamma^\lambda_{\nu\alpha} - G^a_{\lambda\nu}\Gamma^\lambda_{\mu\alpha} + g f_{cb}{}^a A^b_\alpha G^c_{\mu\nu},$$
$$\mathcal{D}_\alpha \psi(x) = (\partial_\alpha + \mathrm{i}e\tilde{A}_\alpha)\psi(x),$$

where $G^a_{\mu\nu}$ is a world tensor with respect to the indices μ, ν, and $SU(n)$-vector with respect to the index a; $\psi(x)$ is a bispinor, and

$$\tilde{A}_\mu(x) = eA_\mu(x) + gT_a A^a_\mu(x) + \frac{1}{2}\hat{I}_{ik}A^{ik}_\mu.$$

In the presence of only an electromagnetic field the derived equations coincide with (11) and (14). The dynamic equations (B.7) and (B.8) can also be found from the Lagrangian

$$L = \sqrt{-g}\left(\frac{1}{2m}\bar{\psi}\overleftarrow{\mathcal{D}}_\mu\overrightarrow{D}^\mu\psi - \frac{1}{2m}\bar{\psi}I_{\mu\nu}\mathcal{F}^{\mu\nu}\psi - \frac{m}{2}\bar{\psi}\psi\right)$$

as a result of the transition to the classical limit (see Sect. 6). This Lagrangian corresponds to the squared Dirac equation in which the external gravitational field and non-Abelian gauge fields are included ($\mathcal{F}^{\mu\nu}$ is defined by (B.4) and (B.5)).

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